

Quantum stochastic processes in two dimensional space-time

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Several stochastic processes with virtual particles in two dimensional space-time are presented whose mean field equations coincide with Schrödinger, Dirac, Klein-Gordon and the quantum mechanic equation for a photon. These processes could be used to detect discrete space-time features at the Planck scale.

Keywords: virtual particles, Schrödinger equation, Dirac equation, Klein-Gordon equation, photon, causality, Planck scale

I. INTRODUCTION

With the appearance of interference effects in the early days of quantum mechanics it became clear that the dynamic of quantum systems could not be simulated by particles with *classical* properties such as definite position and momentum, point interaction, sharp energy, etc. and the *wave* representation was preferred. However with the advent of quantum field theory the particle picture reappeared, not as classical entities, but as virtual particles with ephemeral existence, created and annihilated at all space-time points and with all energy and momentum values not satisfying the usual energy-momentum relations, that is, “*off the mass shell*”. This theory allows for an interpretation of quantum mechanics where the virtual particles are assigned some ontological reality as components building the quantum field[1]. In this interpretation, Feynman diagrams are not only a mathematical term of a perturbation expansion but represent really existent processes.

Historically, virtual particles appeared when the solutions of the dynamic equations of quantum mechanics were expanded in “normal modes” that were interpreted

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as quantum excitations. In this work we take the opposite way, that is, we assume some types of virtual particles with some type of interaction and we show that their collective behaviour is described by the well known dynamic equations of quantum mechanics. For simplicity, and considering possible computer simulations, these “quantum stochastic processes” are studied in two dimensional space-time that include the essential physical features. Furthermore we concentrate on free systems, that is, without external sources.

The usual quantum mechanic equations are presented as arising from different energy-momentum relations, including the less well known quantum mechanic equation for the photon. These are specialized for two dimensional space-time and several stochastic processes are presented whose mean field equations coincide with quantum mechanic ones. Finally some speculations concerning possible interesting non equilibrium effects of the processes are suggested.

II. QUANTUM DYNAMICS FOR FREE SYSTEMS

The energy-momentum relation of a free physical system determines its dynamics because energy and momentum are the generators of space and time translations. For nonrelativistic systems the energy-momentum relation is

$$E = \frac{P^2}{2m} \quad (1)$$

where E is the energy and $P^2 = \mathbf{P} \cdot \mathbf{P}$ is the momentum amplitude squared. In the relativistic case the relation is (in units such that $c = 1$ and with signature $+- - -$)

$$p^\mu p_\mu = E^2 - P^2 = m^2 . \quad (2)$$

Some inconveniences due to the *quadratic* nature of the expression above are avoided by postulating a *linear* energy-momentum relation, that must *also* satisfy the relativistic quadratic relation. This can only be achieved with coefficients γ^μ satisfying a Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. Then we have

$$\gamma^\mu p_\mu = \gamma^0 E - \boldsymbol{\gamma} \cdot \mathbf{P} = m . \quad (3)$$

The relativistic relation (2) is also valid for massless particles, $m = 0$, and in this case the energy becomes a linear dependence with the momentum amplitude

$$E = |\mathbf{P}| . \quad (4)$$

For massive particles, spin \mathbf{S} and momentum \mathbf{P} are decoupled and can have arbitrary directions; however, for particles with zero mass, like the photon, special relativity requires that spin and momentum must be collinear (transversality constraint), that is,

$$\mathbf{S} \times \mathbf{P} = 0 , \quad (5)$$

and the sign of $\mathbf{S} \cdot \mathbf{P}$ corresponds to the positive or negative helicity of the photon. The absolute value in Eq.(4) is problematic when we transform the observables into operators. In order to avoid the absolute value, it is convenient to combine the linear dependence of energy and momentum with the two possible helicity states of the photon by choosing the energy-momentum relation for the photon as[2]

$$E = \mathbf{S} \cdot \mathbf{P} . \quad (6)$$

When this quantity is negative, it must be interpreted as a photon with *positive* energy but with negative helicity.

In quantum physics, the energy and momentum observables are represented by operators in a Hilbert space and when we choose for it the space of square integrable functions, where $E \rightarrow i\partial_t$ and $\mathbf{P} \rightarrow -i\nabla$ (in units such that $\hbar = 1$), we obtain the well known Schrödinger, Klein-Gordon and Dirac equations. In the representation for spin 1 where the spin operators S_l have matrix elements $(S_l)_{jk} = -i\varepsilon_{ljk}$ the quantum mechanics equation for the photon, obtained from Eq.(6), is[2]

$$i\partial_t\Psi_j = \varepsilon_{jlk}\partial_l\Psi_k , \quad (7)$$

where Ψ_k are the tree components of the photon state in the tree dimensional Hilbert space for spin 1. (However, the photon states are restricted to a two dimensional subspace spanned by the helicity ± 1 states.)

Notice that one can *formally* derive Maxwell's equations in vacuum (that is, with vanishing charge and currents) from Eq.(7); however their interpretation is

quite different: the equation above describes the dynamics of a one photon system whereas Maxwell's equations are the evolution equations for a set of observables –the electromagnetic field– of a system made by an indefinite number of photons. Furthermore the alleged derivation is based on the conceptual error of identifying the tree dimensional Hilbert space of spin with the tree dimensional physical space[2].

III. QUANTUM DYNAMICS IN TWO DIMENSIONAL SPACE-TIME

For Schrödinger and Klein-Gordon equations, the specialization for two dimensional space-time (t, x) is straightforward: we just make the association $E \rightarrow i\partial_t$ and $P \rightarrow -i\partial_x$ and we get

$$\partial_t \psi(x, t) = \frac{i}{2m} \partial_x^2 \psi(x, t) , \quad (8)$$

$$(\partial_t^2 - \partial_x^2) \psi(x, t) = -m^2 \psi(x, t) . \quad (9)$$

For Dirac equation we choose the two dimensional matrices and spinor

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad (10)$$

and we become from Eq.(3)

$$\begin{aligned} \partial_t \psi_2(x, t) &= -\partial_x \psi_2(x, t) - i m \psi_1(x, t) \\ \partial_t \psi_1(x, t) &= \partial_x \psi_1(x, t) - i m \psi_2(x, t) . \end{aligned} \quad (11)$$

The quantum equation for the photon in two dimensional space-time is not trivial because the transversality constraint (5) and equations (6) and (7) are essentially in tree dimensional space. We can however make some formal modifications of Eq.(7) in order to get a two dimensional equation. Let us fix the index l at the value $l = 1$ and associate $\partial_1 = \partial_x$. The indices j, k cant take the values 2, 3 and 3, 2 resulting in two coupled equations:

$$\begin{aligned} i\partial_t \Psi_2 &= -\partial_x \Psi_3 , \\ i\partial_t \Psi_3 &= +\partial_x \Psi_2 . \end{aligned} \quad (12)$$

Although the identification of Eq.(7) with Maxwell's equations is questionable, but formally possible, in the two dimensional case we can also obtain the two dimensional Maxwell's equations in vacuum with the misidentification of the photon states with the electric and magnetic fields: $\Psi_2 \rightarrow E$ and $i\Psi_3 \rightarrow B$ obtaining

$$\begin{aligned}\partial_t E &= \partial_x B , \\ \partial_t B &= \partial_x E .\end{aligned}\tag{13}$$

These equations are also obtained by generalizing the four dimensional Maxwell's equations to arbitrary dimension with the techniques of differential geometry and then specializing it to two dimensional space-time[3].

The set of equations (12) can be decoupled with the replacement $\Psi_R = \Psi_2 - i\Psi_3$ and $\Psi_L = \Psi_2 + i\Psi_3$ obtaining

$$\begin{aligned}\partial_t \Psi_R &= -\partial_x \Psi_R , \\ \partial_t \Psi_L &= +\partial_x \Psi_L .\end{aligned}\tag{14}$$

If we try solutions with the form $f(x - vt)$ we can see that Ψ_R and Ψ_L represent a photon moving to the right ($v = 1$) or to the left ($v = -1$) and therefore they are the one dimensional counterparts of the tree dimensional right or left handed photons. Notice that the $+-$ signs in Eq.(12) correspond to different helicity states and in Eq.(14) they correspond to different directions of propagation.

IV. SCHRÖDINGER PROCESS

Let us assume two types of virtual particles A and B in a one dimensional lattice with sites at distance λ . These particles have an asymmetric interaction (violating the action-reaction principle): A particles *reject* B from its location whereas B particles *attract* A to its location. Metaphorically we can say that A *hates* B and B *loves* A or that B particles are *good* and A are *evil*. The rejection of a particle A or B from a site is formalized by the creation of an antiparticle \bar{A} or \bar{B} at the site. Each site can be occupied by any number of particles of type A B or by their corresponding antiparticles \bar{A} \bar{B} . Particles and antiparticles of the same type annihilate in each

site of the lattice leaving only the remaining excess of particles or antiparticles of both types A and B . At each discretized time step, $t \rightarrow t + 1$, corresponding to a time evolution by a small amount τ , each particle of type A creates two antiparticles \bar{B} in the same site and one particle B in each of the two neighbouring sites. This is equivalent to moving one particle B to the right and another to the left. In a similar way, B particles move neighbouring A particles attracting them to its site. This process is realized with some amplitude (probability, if normalized) k .



The same reactions occur exchanging particles and antiparticles.

Before we write the master equation for the time evolution, we can notice some qualitative features of the process. If A rejects B but B attracts A then, transitively, A reject themselves and diffuse. Same conclusion is reached for B . It is obvious that the process has diffusion because in each time step particles and antiparticles are created in neighbouring sites. For instance, if we start with two particles, one A and one B , in one site, after three time steps we will have 110 particles occupying seven sites. It is less obvious that, even though the process has left-right symmetry, we may also have drift to the right or to the left. In order to see how this is possible, recall that A particles *reject* B particles and B *attract* neighbouring A particles. Therefore, if we have an *asymmetric* configuration like AB , the center of the combined distribution will move towards B . The drift direction and velocity is then *encoded* in the shape and relative distribution of both types of particles. It is remarkable that, although the distribution of particles are widely distorted after few time steps, the drift direction and velocity remain invariant (in the continuous limit).

Let $a_s(t)$ and $b_s(t)$ be the number of particles of type A and B respectively at the site s at time t . When $a_s(t)$ or $b_s(t)$ take negative values they denote the number of *antiparticles*. At a particular site of the lattice, the number of particles change as particles or antiparticles are created there by the particles in neighbouring sites.

The time evolution of the process is then defined by the master equations

$$\begin{aligned} a_s(t+1) &= a_s(t) - k (b_{s-1}(t) - 2b_s(t) + b_{s+1}(t)) \\ b_s(t+1) &= b_s(t) + k (a_{s-1}(t) - 2a_s(t) + a_{s+1}(t)) . \end{aligned} \quad (16)$$

Choosing time and space scales such that $\frac{\tau}{\lambda^2} = 1$ we can write these equations as

$$\begin{aligned} \frac{a_s(t+1) - a_s(t)}{\tau} &= -k \frac{b_{s-1}(t) - 2b_s(t) + b_{s+1}(t)}{\lambda^2} \\ \frac{b_s(t+1) - b_s(t)}{\tau} &= k \frac{a_{s-1}(t) - 2a_s(t) + a_{s+1}(t)}{\lambda^2} . \end{aligned} \quad (17)$$

In these equations one can easily recognize the discrete version of the time derivative and of the second space derivative. Taking then the limit $\tau \rightarrow 0$ and $\lambda \rightarrow 0$ with $\frac{\tau}{\lambda^2} \rightarrow 1$ and $k = \frac{1}{2m}$ and also replacing $a_s(t)$ and $b_s(t)$ by continuous functions $a(x, t)$ and $b(x, t)$, the equations above become

$$\begin{aligned} \partial_t a(x, t) &= -\frac{1}{2m} \partial_x^2 b(x, t) \\ \partial_t b(x, t) &= \frac{1}{2m} \partial_x^2 a(x, t) . \end{aligned} \quad (18)$$

Now we can combine the real amplitudes of the virtual particles in one complex field $\psi(x, t) = a(x, t) + ib(x, t)$ and both equations above result in Schrödinger equation given in Eq.(8).

The virtual particles process presented here is a much simplified version of a similar process studied in more details[4].

V. DIRAC PROCESS

Let us assume now four types of virtual particles A, B, C, D in a one dimensional space. These particles always move and create and annihilate each other. Precisely, A and B always move to the left whereas C and D move to the right; D particles create A but A annihilates D and similarly, B particles create C but C annihilates B .

Notice that with regard of the movement, the pair (AB) behaves as chirality opposed to the pair (CD) , that is, we may think of (AB) as having negative helicity (left handed) and (CD) with positive helicity (right handed). Furthermore, with

respect with creation and annihilation, the pair (AC) behaves as the antiparticles of the pair (BD) . Therefore the process globally contains the essence of antimatter but not in an identified way, that is, none of the particles is individually the antiparticle of another. Another interesting qualitative feature is that since D creates A but A annihilates D , transitively, D has self annihilation and the same can be said for all other particles. This self annihilation component is essential to the virtuality of the particle: they can not have permanent existence.

In order to formalize this process let us assume again an infinite one dimensional lattice with sites at distance λ occupied by some number of these particles. Each site can be occupied by any number of particles of type A, B, C, D . At each discretized time step, $t \rightarrow t+1$, corresponding to a time evolution by a small amount τ , particles move one lattice site and are created and destroyed with some amplitude k . In this case we choose space-time units and scale such that $\frac{\tau}{\lambda} = 1$.

Let $a_s(t)$, $b_s(t)$, $c_s(t)$, $d_s(t)$ be the number of particles of type A , B , C , D respectively at the site s at time t . At a particular site of the lattice, the number of particles change as particles migrate, are created or destroyed. The time evolution of the process is then defined by the equations

$$\begin{aligned} a_s(t+1) &= a_{s+1}(t) + kd_s(t) \\ b_s(t+1) &= b_{s+1}(t) - kc_s(t) \\ c_s(t+1) &= c_{s-1}(t) + kb_s(t) \\ d_s(t+1) &= d_{s-1}(t) - ka_s(t) . \end{aligned} \tag{19}$$

We will see that this simple process is described by Dirac equation. Let us first subtract the same quantity from each side of these equations and then use the relation $\frac{\tau}{\lambda} = 1$ to obtain

$$\begin{aligned} \frac{a_s(t+1) - a_s(t)}{\tau} &= \frac{a_{s+1}(t) - a_s(t)}{\lambda} + \frac{k}{\lambda} d_s(t) \\ \frac{b_s(t+1) - b_s(t)}{\tau} &= \frac{b_{s+1}(t) - b_s(t)}{\lambda} - \frac{k}{\lambda} c_s(t) \\ \frac{c_s(t+1) - c_s(t)}{\tau} &= -\frac{c_s(t) - c_{s-1}(t)}{\lambda} + \frac{k}{\lambda} b_s(t) \\ \frac{d_s(t+1) - d_s(t)}{\tau} &= -\frac{d_s(t) - d_{s-1}(t)}{\lambda} - \frac{k}{\lambda} a_s(t) . \end{aligned} \tag{20}$$

In these master equations one can easily recognize the discrete version of the space and time derivatives. Taking then the limit $\tau \rightarrow 0$, $\lambda \rightarrow 0$, $k \rightarrow 0$ with $\frac{\tau}{\lambda} \rightarrow 1$, $\frac{k}{\lambda} \rightarrow m$ and replacing $a_s(t), b_s(t), c_s(t), d_s(t)$ by continuous functions $a(x, t), b(x, t), c(x, t), d(x, t)$, the equations above become

$$\begin{aligned}\partial_t a(x, t) &= \partial_x a(x, t) + m d(x, t) \\ \partial_t b(x, t) &= \partial_x b(x, t) - m c(x, t) \\ \partial_t c(x, t) &= -\partial_x c(x, t) + m b(x, t) \\ \partial_t d(x, t) &= -\partial_x d(x, t) - m a(x, t) .\end{aligned}\tag{21}$$

We can now combine the first two real equations in a complex one by adding the first with the second multiplied by i (and similarly for the last two)

$$\begin{aligned}\partial_t (a(x, t) + i b(x, t)) &= \partial_x (a(x, t) + i b(x, t)) - i m (c(x, t) + i d(x, t)) \\ \partial_t (c(x, t) + i d(x, t)) &= -\partial_x (c(x, t) + i d(x, t)) - i m (a(x, t) + i b(x, t)) ,\end{aligned}\tag{22}$$

and in terms of the complex fields $\psi_1(x, t) = a(x, t) + i b(x, t)$ and $\psi_2(x, t) = c(x, t) + i d(x, t)$, the expression above results in Dirac equation given in (11).

Notice that ψ_1 and ψ_2 have opposed helicity and represent the antimatter of each other.

VI. KLEIN-GORDON PROCESS

After giving two stochastic processes with mean field equations corresponding with Schrödinger and Dirac equations we present now a virtual particle process that results in Klein-Gordon equation (9) arising from the relativistic energy-momentum relation (2).

The second space derivative is obtained in a similar way as in the Schrödinger process in Eq.(15) (but with just one type of diffusing particle) and the rest energy term is also simple, but the *second* time derivative presents some difficulty because, in the discrete time version, it involves *tree* times: $t + 1$, t , $t - 1$. Notice that the Schrödinger and Dirac processes are markovian because the state at time $t + 1$, given by the master equations Eqs.(16,19), are uniquely determined by the state at

time t , without memory on past times. In the present case we must abandon the markovian property because the master equation will also involve the state at time $t - 1$.

Let $a_s(t)$ denote the number of particles (antiparticles, if negative) of type A at site s at time t in a one dimensional lattice with lattice constant λ . At each step τ of discretized time $t \rightarrow t + 1$ all particles reproduce and migrate to neighbouring sites: $\circ A \circ \rightarrow A \circ A$ and are annihilated with probability k . The non-markovian property is that all particles present at time $t - 1$ are removed. The master equation is then

$$a_s(t + 1) = a_{s+1}(t) + a_{s-1}(t) - k a_s(t) - a_s(t - 1) . \quad (23)$$

With some trivial manipulations, and using $\frac{\tau^2}{\lambda^2} = 1$, we can write this equation as

$$\frac{a_s(t - 1) - 2a_s(t) + a_s(t + 1)}{\tau^2} = \frac{a_{s-1}(t) - 2a_s(t) + a_{s+1}(t)}{\lambda^2} - \frac{k}{\lambda^2} a_s(t) \quad (24)$$

Now, taking the limit $\tau \rightarrow 0$, $\lambda \rightarrow 0$, $k \rightarrow 0$ with $\frac{\tau}{\lambda} \rightarrow 1$, $\frac{k}{\lambda^2} \rightarrow m^2$ and replacing $a_s(t)$ by the continuous field $\psi(x, t)$ we obtain Klein-Gordon equation (9).

It is possible to recover the markovian property for a Klein-Gordon process if we make an additional postulate. A process described by $a_s(t)$ can be said to be *causal or markovian* if the state at $t + 1$ is uniquely determined by the state at time t , that is, it satisfies an evolution equation of the type $a_s(t + 1) = F[a_r(t)]$. Now we can define the process to have *reversed causality or reverse markovian* if the state at time $t - 1$, that is, not at future but at past time, is uniquely determined by the state at time t . Such a process satisfies an equation of the type $a_s(t - 1) = F[a_r(t)]$. Notice that the requirement of reversed causality is *different* from time reversal symmetry $T : t \rightarrow -t$. Causality means that the *future* is uniquely determined by the present whereas in reverse causality the *past* is uniquely determined by the present. Let us consider then a markovian Klein-Gordon process described by

$$a_s(t + 1) = \frac{1}{2} a_{s+1}(t) + \frac{1}{2} a_{s-1}(t) - k \frac{1}{2} a_s(t) . \quad (25)$$

This is a process similar to the one of Eq.(23) but *without* the non-markovian term $a_s(t - 1)$. Now, if we require reversed causality we also have

$$a_s(t - 1) = \frac{1}{2} a_{s+1}(t) + \frac{1}{2} a_{s-1}(t) - k \frac{1}{2} a_s(t) . \quad (26)$$

Adding the last two equations we get

$$a_s(t+1) + a_s(t-1) = a_{s+1}(t) + a_{s-1}(t) - ka_s(t) . \quad (27)$$

This is the same master equation (23) for the non-markovian process that in the continuous limit results in Klein-Gordon equation.

Here we have taken a real field but the same could be done with a complex or a many component field in order to account for particles with charges.

VII. PHOTON PROCESS

The photon process is the simplest of all. Let us assume two types of particles R and L . In each time step $t \rightarrow t+1$ of discretized time, corresponding to a time evolution by a small amount τ , R particles move one site to the right and L particles move to the left in a one dimensional lattice with lattice constant λ . The particles can not be at rest and move with constant speed $\frac{\lambda}{\tau}$. Let $r_s(t)$ and $l_s(t)$ denote the number of right and left moving particles at site s at time t . Then, the master equations are simply

$$\begin{aligned} r_s(t+1) &= r_{s-1}(t) \\ l_s(t+1) &= l_{s+1}(t) . \end{aligned} \quad (28)$$

Subtracting $r_s(t)$ [$l_s(t)$] on both sides of the first [second] equation and choosing time and space scales such that $\frac{\lambda}{\tau} = 1$, we can write these equations as

$$\begin{aligned} \frac{r_s(t+1) - r_s(t)}{\tau} &= -\frac{r_s(t) - r_{s-1}(t)}{\lambda} \\ \frac{l_s(t+1) - l_s(t)}{\tau} &= \frac{l_{s+1}(t) - l_s(t)}{\lambda} . \end{aligned} \quad (29)$$

Here we find the discrete version of time and space derivatives. Taking then the limit $\tau \rightarrow 0$ and $\lambda \rightarrow 0$ with $\frac{\tau}{\lambda} \rightarrow 1$ and replacing $r_s(t)$ and $l_s(t)$ by continuous functions $\Psi_R(x, t)$ and $\Psi_L(x, t)$ we obtain the quantum mechanic dynamic equation for a photon given in Eq.(14).

VIII. CONCLUSION

In a way consistent with the quantum field theory interpretation of quantum mechanics we have found stochastic processes with virtual particles whose mean field equations correspond to Schrödinger, Dirac, and Klein-Gordon equations, as well as the corresponding quantum mechanic equation for the photon.

One possible interest of studying these “quantum stochastic processes” appears if we consider that the mean field equations are obtained when we take the limit from a discrete to a continuous space-time. However it is well known in non-equilibrium statistical mechanics that there are some features of discrete space-time processes that could appear in a computer simulation that are not present in the solutions of the mean field equations. This could be interesting in order to predict some effects due to an inherent discreteness of space-time of the order of Planck time $T_P \approx 10^{-42}s$ and length $L_P \approx 10^{-35}m$ as is suggested in some quantum gravity theories. Such simulations with $\tau \approx L_P$ and $\lambda \approx L_T$ are beyond reach with today computing power, however with the exponentially increasing computing capabilities it may sometime be possible.

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